

Closed manifolds admitting metrics with the same geodesics

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Abstract

The goal of this survey is to give a list of recent results about topology of manifolds admitting different metrics with the same geodesics. We emphasize the role of the theory of integrable systems in obtaining these results.

1 Introduction

1.1 Definitions

Definition 1 *Let g be a Riemannian metric on a manifold M^n of dimension $n \geq 2$. A Riemannian metric \bar{g} on M^n is called **geodesically equivalent** to g , if every geodesic of \bar{g} , considered as an unparameterized curve, is a geodesic of g .*

Trivial examples of geodesically equivalent metrics can be obtained by considering proportional metrics g and $C \cdot g$, where C is a positive constant.

Definition 2 *A manifold M^n is called **geodesically rigid**, if every two geodesically equivalent Riemannian metrics on M^n are proportional.*

In other words, on geodesically rigid manifolds, unparameterized geodesics define the metric (modulo multiplication by a constant).

1.2 History

The theory of geodesically equivalent metrics has a long and fascinating history that goes back to the works of Beltrami, Dini and Levi-Civita.

Beltrami [2] was the first to observe that two nonproportional metrics (even on closed manifolds) can have the same geodesics.

At the end of his paper [2], Beltrami formulated the problem of describing all geodesically equivalent metrics (for surfaces.) It is not clear from the text

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whether he assumed the local or the global description; actually, his motivation came from a certain problem of cartography, which requires the global setting. Nevertheless, partially because of strong results of Dini, Levi-Civita, Weyl, E. Cartan and Eisenhart, the theory of geodesically equivalent metrics was mostly a local geometry.

Locally, in a neighborhood of almost every point, a complete description of geodesically equivalent metrics has been given by Dini [13] for surfaces and Levi-Civita [18] for manifolds of arbitrary dimension. As a corollary of this description, one can show that, at least for dimensions two and three, every open manifold has non-proportional geodesically equivalent metrics.

Later, geodesically equivalent metrics were considered by Weyl, E. Cartan and Eisenhart. Weyl studied geodesically equivalent metrics on the tensor level and found a few tensor reformulations of geodesic equivalence. One of his most remarkable results is the construction [35] of the projective Weyl tensor: if two metrics are geodesically equivalent, then their projective Weyl tensors coincide. E. Cartan [11] studied geodesic equivalence on the level of affine connections. He introduced the so-called projective connection, which allows reconstruction of geodesics as unparameterized curves. In his book [14], Eisenhart systematically applied both methods and obtained a series of local results.

It is clear that the classics such as Lie [19], Painlevé [30], Levi-Civita and Eisenhart understood well the connection between integrable systems and geodesically equivalent metrics. But they did not use it, probably because they mostly were interested in the local aspects of geodesically equivalent metrics.

Global aspects have been intensively studied since 50th, firstly by French (Lichnerowicz), Soviet (Rashevskii and Solodovnikov) and Japanese (Yano) geometry schools. But, probably because of the influence of earlier researcher, all known global results require fairly strong additional geometrical assumptions.

Roughly speaking, one takes some geometric assumption written in tensor form, combines it with one of the tensor reformulations of geodesic equivalence and deduces some new object with global geometric properties, see the surveys of Mikes [29] and Aminova [1].

Recently, it was found that integrable systems provides a very effective tool for understanding the topology of the manifolds admitting geodesically equivalent metrics. We describe the connection between geodesically equivalent metrics and integrable geodesic flows in Section 3.1. Roughly speaking, the existence of \bar{g} geodesically equivalent to g allows one to construct integrals for the geodesic flow of g , see Theorem 4 for details. If the metric g and \bar{g} are strictly-non-proportional, the geodesic flow of g is Liouville-integrable.

2 Resent results

Theorem 1 ([27]) *Let M^n be a closed connected manifold. Suppose two non-proportional Riemannian metrics g, \bar{g} on M^n are geodesically equivalent. If the fundamental group of M^n is infinite, then there exist $r \in \{1, 2, \dots, n-1\}$, a Riemannian metric \tilde{g} and foliations B_r (of dimension r) and B_{n-r} (of dimension*

$n - r$) such that, in a neighborhood $U(p)$ of every point $p \in M^n$, there exist coordinates

$$(\bar{x}, \bar{y}) = ((x_1, x_2, \dots, x_r), (y_{r+1}, y_{r+2}, \dots, y_n))$$

such that the x -coordinates are constant on every fiber of the foliation $B_{n-r} \cap U(p)$, the y -coordinates are constant on every fiber of the foliation $B_r \cap U(p)$, and the metric \tilde{g} has the block-diagonal form

$$ds^2 = \sum_{i,j=1}^r G_{ij}(\bar{x}) dx_i dx_j + \sum_{i,j=r+1}^n H_{ij}(\bar{y}) dy_i dy_j, \quad (1)$$

where the first block depends on the first r coordinates and the second block depends on the remaining $n - r$ coordinates.

Theorem 1 already gives us the complete list of all geodesically rigid closed surfaces [21]: a closed connected surface is geodesically rigid if and only if its Euler characteristic is negative.

More precisely, a closed connected surface of negative Euler characteristic has infinite fundamental group, and admits no one-dimensional foliation, so it is geodesically rigid.

A closed connected surface of nonnegative Euler characteristic is not geodesically rigid: it is diffeomorphic to the sphere or the projective plane or the torus or the Klein bottle. Examples of nonproportional geodesically equivalent metrics on the sphere and on the projective plane were essentially constructed by Beltrami [2]. Since the geodesics of every flat metric are straight lines, every two flat metrics on the torus (or on the Klein bottle) are geodesically related (that is, there exists a diffeomorphism that takes the geodesics of the first metric to the geodesics of the second), so the torus or the Klein bottle are not geodesically rigid as well.

For dimension three, a direct corollary [27] of Theorem 1 is

Corollary 1 *Suppose M^3 is a connected closed manifold. Suppose there exist nonproportional Riemannian metrics on M^3 that are geodesically equivalent. Then, modulo the Poincaré conjecture, M^3 is finitely covered by the sphere S^3 or by the product $F^2 \times S^1$, where F^2 is a closed surface.*

It appears that Corollary 1 is true [25, 28] also without assuming the Poincaré conjecture:

Theorem 2 *Let nonproportional Riemannian metrics g and \bar{g} be geodesically equivalent on a closed connected three-dimensional manifold M^3 . Then the manifold is homeomorphic either to a lens space or to a Seifert manifold with zero Euler number. Every lens space and every Seifert manifold with zero Euler number admits geodesically equivalent metrics which are nonproportional.*

Theorems 1,2 give us a complete list of closed connected manifolds of dimension two and three admitting nontrivial geodesic equivalence. It will be much

more complicated to obtain such list in every dimension. But still, in every dimension $n \geq 2$, there exists infinitely many geodesically rigid manifolds [23, 27]:

Theorem 3 *Every closed connected manifold admitting a Riemannian metric of negative sectional curvature is geodesically rigid.*

Note that in view of result of Borel [10], in every dimension there exist infinitely many closed manifolds admitting metrics of negative sectional curvature.

3 Methods and ideas

3.1 Integrability for the geodesic flows of geodesically equivalent metrics

New methods for the global (= on closed or complete manifolds) investigation of geodesically equivalent metrics are based on the following observation [20, 21, 26]: the existence of \bar{g} geodesically equivalent to g allows one to construct commuting integrals for the geodesic flow of g .

Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be Riemannian metrics on a manifold M^n . Consider the (1,1)-tensor L given by the formula

$$L_j^i \stackrel{\text{def}}{=} \left(\frac{\det(\bar{g})}{\det(g)} \right)^{\frac{1}{n+1}} \bar{g}^{i\alpha} g_{\alpha j}. \quad (2)$$

Then, L determines the family S_t , $t \in R$, of (1,1)-tensors

$$S_t \stackrel{\text{def}}{=} \det(L - t \text{Id}) (L - t \text{Id})^{-1}. \quad (3)$$

Remark 1 *Although $(L - t \text{Id})^{-1}$ is not defined for t lying in the spectrum of L , the tensor S_t is well-defined for every t . Moreover, S_t is a polynomial in t of degree $n - 1$ with coefficients being (1,1)-tensors.*

We will identify the tangent and cotangent bundles of M^n by g . This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n .

Theorem 4 *If g, \bar{g} are geodesically equivalent, then, for every $t_1, t_2 \in R$, the functions*

$$I_{t_i} : TM^n \rightarrow R, \quad I_{t_i}(v) \stackrel{\text{def}}{=} g(S_{t_i}(v), v) \quad (4)$$

are commuting integrals for the geodesic flow of g .

In other direction these theorem is wrong; a counterexample could be found in [22].

Theorem 5 ([22]) *Suppose for every $t \in \mathbb{R}$ the function I_t given by (4) is an integral for the geodesic flow of g . If the Nijenhuis torsion N_L vanishes, the metrics are geodesically equivalent.*

Theorem 6 ([9]) *Let g, \bar{g} be geodesically equivalent. Then the Nijenhuis torsion N_L vanishes.*

3.2 What is special in these integrals?

Definition 3 *Two metrics g and \bar{g} are **strictly-non-proportional** at $x \in M^n$, if all roots of $P(t) := \det(g - t\bar{g})$ are simple.*

Let us assume that there exists a point where the metrics are strictly-non-proportional. Then, it is so [27] at almost every point, and the family I_t contains n integrals that are functionally independent almost everywhere. Hence, the geodesic flow of the metric is Liouville-integrable.

Let us note that

- the integrals are quadratic in velocities,
- at every point $x \in M^n$, the integrals (considered as quadratic forms) can be simultaneously diagonalizable.

Integrable systems with such properties [32] are known as Stäckel systems. Locally, in a given coordinate system, it can be defined by using a $(n \times n)$ -matrix such that its columns depend on the corresponding coordinate only.

Not every stäckel system can come from geodesically equivalent metrics. The additional assumption is that the stäckel matrix can be chosen to be a Vandermonde matrix.

Stäckel systems with such property were also intensively studied. One of the reasons for it that they satisfy Robertson's [31] condition, which imply that its quantization is quantum-integrable as well. The second reasons is that all stäckel systems coming from physics are of this type (or, a degeneration of systems of this type).

It is possible to show [22], that this extra-condition is also a sufficient condition for the existence of geodesically equivalent metrics.

The systems with this condition appear independently and under different names (L-systems, Benenti systems, quasi-bihamiltonian systems) in works of different authors [3, 4, 5, 6, 16, 12].

3.3 If the metrics are strictly non-proportional

Theorem 4 can be used most efficiently when there exists a point of a manifold where the metrics are strictly-non-proportional. Then, the geodesic flow of g is Liouville-integrable, and we can apply the well-developed machinery of integrable systems. For example, the following theorem follows directly from Theorem 4 and Taimanov [33].

Theorem 7 *Suppose M^n is a connected closed manifold. Let the real-analytic Riemannian metrics g and \bar{g} on M^n be geodesically equivalent. Suppose there exists a point of the manifold where the metrics are strictly-non-proportional. Then, the following statements hold.*

1. *The first Betti number $b_1(M^n)$ is not greater than n .*
2. *The fundamental group $\pi_1(M^n)$ is virtually Abelian.*

The integrals are quadratic in velocities. Combining this fact with topological obstructions [17] for the existence of quadratically-integrable geodesic flows on closed surfaces, we obtain Theorem 3 for dimension two [21].

Note that, in view of results [34], Corollary 1 follows from Theorem 7 under the additional assumption that the metrics are real-analytic and that there exists a point where the metrics are strictly-non-proportional .

3.4 Geodesic equivalence and zero entropy

All results of this section are joint with Kruglikov; the proofs will be published elsewhere.

It is expected, that an integrable geodesic flow has zero topological entropy. This is not always the case. There are examples [7, 8] of integrable flows with non-zero entropy. But still it is possible to show that if the integrals come from strictly-non-proportional geodesically equivalent metrics by applying Theorem 4, the topological entropy of the geodesic flow must be zero.

Theorem 8 *Suppose the Riemannian metrics g, \bar{g} on a closed connected M^n are geodesically equivalent. Suppose there exists a point where the metrics are strictly non-proportional. Then, the topological entropy of the geodesic flow of g is zero.*

Combining this theorem with the famous Yomdin's Theorem [15], we obtain

Corollary 2 *Suppose the Riemannian metrics g, \bar{g} on a closed connected M^n are geodesically equivalent. Suppose there exists a point where the metrics are strictly non-proportional. Then, the manifold is finitely covered by the product of a rational-elliptic manifold and the torus.*

3.5 General case

We will sketch the proof of Theorem 1. Consider the $(1,1)$ -tensor L given by the formula (2). All its eigenvalues are real. At every point $x \in M^n$, let us denote them by $\lambda_1(x) \leq \dots \leq \lambda_n(x)$. It appears that they are globally ordered [26, 27, 28]:

Theorem 9 *Let (M^n, g) be a connected Riemannian manifold. Suppose every two points of the manifold can be connected by a geodesic. Let Riemannian metric \bar{g} on M^n be geodesically equivalent to g .*

Then, for every $i \in \{1, \dots, n-1\}$, for all $x, y \in M^n$, the following holds:

1. $\lambda_i(x) \leq \lambda_{i+1}(y)$.
2. If $\lambda_i(x) < \lambda_{i+1}(x)$ for some $x \in M^n$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.
3. If $\lambda_i(x) = \lambda_{i+1}(y)$, then there exists $z \in M^n$ such that $\lambda_i(z) = \lambda_{i+1}(z)$.

Thus, if g and \bar{g} on closed connected M^n are geodesically equivalent, then the following two cases are possible:

Case 1: There exists $r \in \{1, \dots, n-1\}$ and a constant $\lambda \in \mathbb{R}$ such that, for every $x \in M^n$

$$\lambda_r(x) < \lambda < \lambda_{r+1}(x).$$

Case 2: The following two conditions hold:

- (i) For every $r \in \{1, \dots, n-1\}$, the maximum $\max_{x \in M^n}(\lambda_r(x))$ is equal to the minimum $\min_{x \in M^n}(\lambda_{r+1}(x))$.
- (ii) At least one of the eigenvalues of L is not constant.

In the first case, it is possible to canonically construct [27] the metric \tilde{g} and the foliations B_r and B_{n-r} as in Theorem 1.

Let us explain where the foliations are coming from. At every point $x \in M^n$, let us denote by $V_r(x)$ ($V_{n-r}(x)$, respectively) the sum

$$\bigoplus_{i=1}^r E_{\lambda_i} \quad \left(\bigoplus_{i=r+1}^n E_{\lambda_i}, \text{ respectively} \right)$$

where $E_{\lambda_i} \in T_x M^n$ is the eigenspace corresponding to λ_i .

Under the assumptions of Case 1, V_r and V_{n-r} are smooth distributions of dimensions r and $n-r$. By Theorem 6, they are integrable. Then, they generate two foliations B_r and B_{n-r} .

The construction of the metric \tilde{g} from Theorem 1 is based on the classical Levi-Civita's Theorem [18].

In the second case, it is possible to show that the fundamental group of the manifold is finite. The key instrument for it is Theorem 6 of the paper [27] which, roughly speaking, tells us that every (closed) manifold with two geodesically equivalent metrics satisfying (i), (ii) has a closed submanifold U with two geodesically equivalent metrics satisfying (i), (ii) such that the natural homomorphism $\text{Id}_* : \pi_1(U) \rightarrow \pi_1(M^n)$ is a surjection. Consequently applying this theorem, we come to one of the following subcases:

SC 1: The dimension n of the manifold M^n is $q+1$, where $q \geq 1$. The eigenvalues $\lambda_1 = \dots = \lambda_q \stackrel{\text{def}}{=} \lambda$ are constant, the eigenvalue λ_{q+1} is not constant and there exists $z \in M^{q+1}$ such that $\lambda_{q+1}(z) = \lambda$.

SC 2: The dimension n of the manifold M^n is 2. The eigenvalues λ_1 and λ_2 are not constant and there exists a point $z \in M^2$ such that $\lambda_1(z) = \lambda_2(z)$.

- SC 3:** The dimension n of the manifold M^n is $q + 2$, where $q \geq 1$, the eigenvalues λ_1 and λ_{q+2} are not constant and there exist $z_1, z_2 \in M^n$ such that $\lambda_1(z_1) = \lambda_{q+2}(z_2)$.
- SC 4:** The dimension n of the manifold M^n is $n = q + r + 1$; $q > 0, r > 0$. The eigenvalues $\lambda_1 = \lambda_2 = \dots = \lambda_r$ and $\lambda_{r+2} = \lambda_{r+3} = \dots = \lambda_n$ are constant. The eigenvalue λ_{r+1} is not constant. There exist points $z_0, z_1 \in M^n$ such that $\lambda_{r+1}(z_0) = \lambda_1$ and $\lambda_{r+1}(z_1) = \lambda_n$.

It is possible to show [27] that in all four subcases the fundamental group is finite.

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